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GAUGING WESS-ZUMINO-WITTEN MODELS [1]

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ABSTRACT

We review some aspects of gauged WZW models. By choosing a solvable subgroup as gauge group, one is lead to three main applications: the construction of field theories with an extended conformal symmetry, the construction of the effective action of (extended) 2D gravities and the systematic construction of string theories with some extended gauge symmetry.

1. Introduction

For any semi-simple (super) Lie algebra $\mathcal{G}$, one can write down the corresponding Wess-Zumino-Witten model [1]. Its action is given by $\mathcal{S} = \kappa\mathcal{S}^+$ with κ the level and $\mathcal{S}^+$ given by

$$S^+[g] = \frac{1}{4\pi x}\int d^2z\; str\left\{\partial g^{-1}\bar{\partial}g\right\} + \frac{1}{12\pi x}\int d^3z\; \varepsilon^{\alpha\beta\gamma}\, str\left\{g_{,\alpha}g^{-1}g_{,\beta}g^{-1}g_{,\gamma}g^{-1}\right\}, \quad (1.1)$$

with x the index of the representation. The functional $\mathcal{S}^+$ satisfies the Polyakov-Wiegman identity [2]:

$$S^+[hg] = S^+[h] + S^+[g] - \frac{1}{2\pi x}\int str\Big(h^{-1}\partial h\bar{\partial}gg^{-1}\Big). \quad (1.2)$$

We will also use the functional $S^-[g]$, defined by $S^-[g] = S^+[g^{-1}]$. Using the equations of motion of the WZW model,

$$\delta S^+[g] = \frac{1}{2\pi x}\int str\left\{\bar{\partial}(g^{-1}\partial g)g^{-1}\delta g\right\} = \frac{1}{2\pi x}\int str\left\{\partial(\bar{\partial}gg^{-1})\delta gg^{-1}\right\}, \quad (1.3)$$

one gets the conserved affine currents

$$J_z = -\frac{\kappa}{2}g^{-1}\partial g, \qquad J_{\bar{z}} = \frac{\kappa}{2}\bar{\partial}gg^{-1}. \quad (1.4)$$

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The OPE's of the affine currents close:

$$J^a_z(z)J^b_z(w) = -\frac{\kappa}{2}g^{ab}(z-w)^{-2} + (z-w)^{-1}f^{abc}J_{z\,c}(w) + \cdots, \tag{1.5}$$

where we defined the Killing metric by $str(t_a t_b) = -x\, g_{ab}$. A similar relation holds for $J_{\bar{z}}$. The energy-momentum tensor of the WZW model is given by the Sugawara construction

$$T = \frac{1}{x\left(\kappa + \tilde{h}\right)} str J_z J_z, \tag{1.6}$$

with $\tilde{h}$ the dual Coxeter number and it satisfies the Virasoro algebra with the central extension given by:

$$c = \frac{\kappa(d_B - d_F)}{\kappa + \tilde{h}}, \tag{1.7}$$

where d_B, d_F resp., is the number of bosonic, fermionic resp., generators of the (super) Lie algebra.

Not only does the WZW model provide us with a large class of non-trivial conformal field theories (cft's), it can also be used to construct other cft's. The way to achieve this goes through gauging the WZW model. Using the Polyakov-Wiegman identity, it is straightforward to gauge any subalgebra of $\mathcal{G}$. Indeed, $\kappa S^-[h_+ g h_-]$, with $h_\pm \in \mathcal{H}_\pm \subseteq \mathcal{G}$, can be worked out using eq. (1.2) and the result is clearly invariant under gauge transformations

$$g \to \gamma_+^{-1} g \gamma_-^{-1}, \qquad h_+ \to h_+\gamma_+, \qquad h_- \to \gamma_- h_-. \tag{1.8}$$

However, if one tries to write out the action in terms of the gauge fields $A_z = \partial h_- h_-^{-1}$ and $A_{\bar{z}} = -h_+^{-1}\bar{\partial} h_+$, one generically obtains a non-local expression. A well-known way out is taking $\mathcal{H}_+ = \mathcal{H}_-$ and choosing $\kappa(S^-[h_+ g h_-] - S^-[h_+ h_-])$ as the gauge invariant action. Another possibility is choosing the subgroup to be such that all non-local terms vanish. Precisely the last case will be studied in this paper.

2. sl(2) embeddings and extended conformal symmetries

We consider a non-trivial embedding of $sl(2)$ into a (super) Lie algebra $\mathcal{G}$. The adjoint representation of $\mathcal{G}$ decomposes into $sl(2)$ irreps:

$$\text{adjoint}(\mathcal{G}) = \bigoplus_{j \in \frac{1}{2}\mathbf{N}} n_j\, \underline{2j+1}, \tag{2.1}$$

with $n_j \in \mathbf{N}$, the multiplicities. The $sl(2)$ embedding induces a natural grading on $\mathcal{G}$ given[2] by the eigenvalues of e_0. In [3], it was shown that constraining the affine

[2] The $sl(2)$ generators are denoted by $\{e_{\ddagger}, e_{=}, e_0\}$ and they satisfy $[e_0, e_{\ddagger}] = 2e_{\ddagger}$, $[e_0, e_{=}] = -2e_{=}$ and $[e_+, e_-] = e_0$.

currents of the coresponding WZW model as:

$$J = T + \frac{\kappa}{2} e_{=}, \tag{2.2}$$

where J stands for J_z and $T \in \ker \mathrm{ad}_{e_{\neq}} \mathcal{G}$, breaks the affine symmetry down to some extension of the Virasoro symmetry which is generated by n_j currents of conformal dimension $j+1$ for $j \in \frac{1}{2}\mathbf{N}$. In order to realize this in terms of an action, we consider a WZW model on $\mathcal{G}$ where we gauge the subalgebra $\Pi_{>0}\mathcal{G}$. We use the notation that for $X \in \mathcal{G}$, $\Pi_{>0}X$ projects out the strict positively graded part of X, with the grading given by the $sl(2)$ induced grading. The gauge invariant action is

$$\mathcal{S} = \kappa S^{-}[g] + \frac{1}{\pi x} \int str\, A\, J, \tag{2.3}$$

where A stands for $A_{\bar{z}}$ and $A \in \Pi_{>0}\mathcal{G}$. However, this action has no chance of reproducing eq. (2.2), as the equation of motion for A puts $\Pi_{<0}J$ to zero. Instead, we take [4]:

$$\mathcal{S} = \kappa S^{-}[g] + \frac{1}{\pi x} \int str\, A(J - \frac{\kappa}{2} e_{=} - \frac{\kappa}{2}[e_{=}, \tau]) - \frac{\kappa}{4\pi x} \int str[e_{=}, \tau]\bar{\partial}\tau, \tag{2.4}$$

where $\tau \in \Pi_{1/2}\mathcal{G}$ was introduced in order to obtain a gauge invariant action. One checks that eq. (2.4) is invariant under:

$$\delta g = \eta g, \qquad \delta A = \bar{\partial}\eta + [\eta, A], \qquad \delta\tau = -\Pi_{\frac{1}{2}}\eta, \tag{2.5}$$

and $\eta \in \Pi_{>0}\mathcal{G}$. One can verify that a combination of the equation of motion of A and a gauge choice, precisely reproduces eq. (2.2).

Once a Lagrangian formulation has been obtained, we can follow the standard procedure to quantize this system. Taking $A = 0$ as the gauge choice and introducing ghosts $c \in \Pi_{>0}\mathcal{G}$ and anti-ghosts $b \in \Pi_{<0}\mathcal{G}$, we get the gauge fixed action:

$$\mathcal{S}_{\rm gf} = \kappa S^{-}[g] + \frac{\kappa}{4\pi x} \int str[\tau, e_{=}]\bar{\partial}\tau + \frac{1}{2\pi x} \int str\, b\bar{\partial}c, \tag{2.6}$$

and the nilpotent BRST charge $\mathcal{Q}_{HR}$:

$$\mathcal{Q}_{HR} = \frac{1}{4\pi i x} \oint str \left\{ c \left(J - \frac{\kappa}{2} e_{=} - \frac{\kappa}{2}[e_{=}, \tau] + \frac{1}{2} J^{\rm gh} \right) \right\}. \tag{2.7}$$

where $J^{\rm gh} = \frac{1}{2}\{b, c\}$. The generators of the extended conformal algebra are the generators of the cohomology of $\mathcal{Q}_{HR}$ computed on the algebra $\mathcal{A}$ which is generated by $\{b, \hat{J}, \tau, c\}$, with $\hat{J} = J + J^{\rm gh}$, and consisting of all regularized products of the generating fields and their derivatives modulo the usual relations between different orderings, derivatives, etc. The calculation of this cohomology was performed in [4, 5] heavily using spectral sequence techniques. We summarize the main results.

1. The cohomology is only non-trivial at ghostnumber 0.

2. For every $sl(2)$ irrep in the decomposition eq. (2.1), we obtain a conformal current T^{j,α_j}, $\alpha_j \in \{1,\cdots,n_j\}$. They have the form:

$$T^{j,\alpha_j} = \sum_{n=0}^{2j} T^{j,\alpha_j}_{j-\frac{n}{2}} \tag{2.8}$$

   where each term in the sum has definite grading $j - n/2$ (τ has been assigned grading 0) and the leading term T^{j,α_j}_j is proportional to the appropriate element of $\Pi_{\ker ad e_{\ddagger}}(\hat{J} + (\kappa/4)[\tau,[e_{=},\tau]])$. The other terms are recursively determined from this.

3. The Virasoro subalgebra is associated to the embedded $sl(2)$ itself. The other currents T^{j,α_j} are primary fields of conformal dimension $j+1$.

4. The central charge of the system is given by

$$c = \frac{1}{2}c_{\rm crit} - \frac{(d_B - d_F)\tilde{h}}{\kappa + \tilde{h}} - 6y(\kappa + \tilde{h}), \tag{2.9}$$

   where $c_{\rm crit}$ is the critical central charge and y the index of embedding.

5. The map $T^{j,\alpha_j} \rightarrow T^{j,\alpha_j}_0$ is an algebra isomorphism. It is the quantum Miura transformation.

Concluding, we see that we obtained a very systematic way to study extended conformal symmetries. Though not all extensions of the Virasoro algebra can be obtained this way, a very large class is covered. We end this section with an example.

The super Lie algebra $sl(2|1)$ is generated by a bosonic $su(2) \oplus u(1)$ sector: $\{e_{\ddagger}, e_{=}, e_0, u_0\}$ with $[e_0, e_{\ddagger}] = +2e_{\ddagger}$, $[e_0, e_{=}] = -2e_{=}$ and $[e_{\ddagger}, e_{=}] = e_0$. The fermionic generators, $g_{\pm}$, $\bar{g}_{\pm}$ are $sl(2)$ doublets, while $g_{\pm}$ ($\bar{g}_{\pm}$) have eigenvalue $+1$ (-1) under ad_{u_0}. The remaining commutation relations are easily derived from the 3×3 matrix representation $e_{\ddagger} = E_{12}$, $e_{=} = E_{21}$, $e_0 = E_{11} - E_{22}$, $u_0 = -E_{11} - E_{22} - 2E_{33}$, $g_+ = E_{13}$, $g_- = E_{23}$, $\bar{g}_+ = E_{32}$ and $\bar{g}_- = E_{31}$. The index of the fundamental representation is $x = 1/2$. Only one non-trivial $sl(2)$ embedding is possible. The WZW model, with action $\kappa S^-[g]$ on $sl(2|1)$ gives rise to affine currents $J = E^{\ddagger}e_{\ddagger} + E^0 e_0 + E^{=}e_{=} + U^0 u_0 + F^+ g_+ + F^- g_- + \bar{F}^+ \bar{g}_+ + \bar{F}^- \bar{g}_-$.

We now follow the general strategy outlined above. Using the canonical grading induced by the $sl(2)$ embedding we get for the affine currents:

	$E^{\ddagger}$	E^0	$E^{=}$	U^0	F^+	F^-	$\bar{F}^+$	$\bar{F}^-$
grade	1	0	-1	0	$1/2$	$-1/2$	$1/2$	$-1/2$

.

The gauge group is generated by $\{e_{\ddagger}, g_+, \bar{g}_+\}$. Taking $A = A^{\ddagger}e_{\ddagger} + A^+g_+ + \bar{A}^+\bar{g}_+$, and $\tau = \tau g_+ + \bar{\tau}\bar{g}_+$, we get from eq. (2.4) the invariant action. The quantization yields 4 generators for the BRST cohomology:

$$
\begin{aligned}
T_{N=2} &= \frac{2\kappa}{\kappa+1}\left(\hat{E}^{\ddagger} + \hat{\bar{F}}^{+}\tau + \hat{F}^{+}\bar{\tau} - \frac{2}{\kappa}\hat{U}^0\hat{U}^0 + \frac{2}{\kappa}\hat{E}^0\hat{E}^0\right. \\
&\qquad \left. -\frac{\kappa+1}{\kappa}\partial\hat{E}^0 - \frac{\kappa+1}{4}\left(\tau\partial\bar{\tau} - \partial\tau\bar{\tau}\right)\right) \\
G_+ &= \sqrt{\frac{4\kappa}{\kappa+1}}\left(\hat{F}^+ - \tau\left(\hat{E}^0 + \hat{U}^0\right) + \frac{\kappa+1}{2}\partial\tau\right) \\
G_- &= \sqrt{\frac{4\kappa}{\kappa+1}}\left(\hat{\bar{F}}^{+} - \bar{\tau}\left(\hat{E}^0 - \hat{U}^0\right) + \frac{\kappa+1}{2}\partial\bar{\tau}\right) \\
U &= -4\left(\hat{U}^0 - \frac{\kappa}{4}\tau\bar{\tau}\right),
\end{aligned}
\tag{2.10}
$$

which satisfies the $N = 2$ superconformal algebra with $c_{N=2} = -3(1+2\kappa)$ (this follows from eq. (2.9), with $y = 1$, $\tilde{h} = 1$ and $c_{crit} = 6$). The Miura transform yields the standard free field realization of the $N = 2$ superconformal algebra:

$$
\begin{aligned}
T_{N=2} &= \partial\varphi\partial\bar{\varphi} - \frac{\sqrt{\kappa+1}}{2}\partial^2(\varphi+\bar{\varphi}) - \frac{1}{2}(\psi\partial\bar{\psi} - \partial\psi\bar{\psi}), \\
G_+ &= -\psi\partial\varphi + \sqrt{\kappa+1}\partial\psi, \qquad G_- = -\bar{\psi}\partial\varphi + \sqrt{\kappa+1}\partial\bar{\psi} \\
U &= \psi\bar{\psi} - \sqrt{\kappa+1}(\partial\varphi - \partial\bar{\varphi}),
\end{aligned}
\tag{2.11}
$$

where $\partial\varphi(z_1)\partial\bar{\varphi}(z_2) = z_{12}^{-2}$, $\psi(z_1)\bar{\psi}(z_2) = z_{12}^{-1}$ and we introduced some simplifying rescalings: $\partial\varphi = 2(\kappa+1)^{-1/2}(\hat{E}^0 + \hat{U}^0)$, $\psi = \sqrt{\kappa}\tau$, $\partial\bar{\varphi} = 2(\kappa+1)^{-1/2}(\hat{E}^0 - \hat{U}^0)$ and $\bar{\psi} = \sqrt{\kappa}\bar{\tau}$.

3. Effective actions for 2D gravity

Consider a cft whose fields we collectively denote by ϕ and which is described by an action $\mathcal{S}[\phi]$. The model has an extended conformal algebra generated by the currents $T_i[\phi]$, with $i \in \{1, \cdots, N\}$. The induced action for the extended gravity theory is, in the light-cone gauge, given by

$$
e^{-\Gamma[\mu]} = \int [d\phi] e^{-\Gamma[\phi] - \frac{1}{\pi}\sum_{i=1}^{N}\int \mu^i T_i[\phi]},
\tag{3.1}
$$

where the μ^i are the generalized Beltrami differentials. The effective action for the $2D$ extended gravity theory is then

$$
e^{-W[\check{T}]} = \int [d\mu] e^{-\Gamma[\mu] + \frac{1}{\pi}\sum_{i=1}^{N}\int \mu^i \check{T}_i}.
\tag{3.2}
$$

Reversing the order of integration, we obtain

$$e^{-W[\check{T}]} = \int [d\phi] \prod_{i=1}^{N} \delta(\check{T} - T[\phi]) e^{-\Gamma[\phi]}, \tag{3.3}$$

which is extremely hard to compute due to the non-trivial Jacobian.

However, in previous section, we considered a particular form for the cft described by $\mathcal{S}[\phi]$ which will allow for the computation of the Jacobian [4, 6]. We get for the effective action:

$$\begin{aligned} e^{-W[\check{T}]} &= \int [\delta g g^{-1}][d\tau][dA][d\mu] \left(\mathrm{Vol}\left(\Pi_{>0}\mathcal{G}\right)\right)^{-1} \\ &\quad \exp\left(-\mathcal{S} - \frac{1}{\pi} \sum_{j\in\frac{1}{2}\mathbf{N}} \sum_{\alpha_j=1}^{n_j} \int \mu_{j,\alpha_j} \left(T^{j,\alpha_j} - \check{T}^{j,\alpha_j} \right) \right), \end{aligned} \tag{3.4}$$

where $\mathcal{S}$ was given in eq. (2.4). In order to compute the effective action, we choose the highest weight gauge: $\tau = \Pi_{>0}[e_{\mp}, J_z] = 0$ and we find using $[\delta g g^{-1}] = [dJ] \exp\left(-2\tilde{h} S^{-}[g]\right)$

$$W[\check{T}] = \kappa_c S_{-}[g'] \tag{3.5}$$

where $\kappa_c = \kappa + 2\tilde{h}$ and[3] $\partial g' g'^{-1} = e_{=} + \sum_{j,\alpha_j} C(j,\alpha_j) \check{T}^{j,\alpha_j} t_{j,\alpha_j}$. From eq. (2.9) we get the level as a function of the central charge:

$$12 y \kappa_c = 12 y \tilde{h} - \left(c - \frac{1}{2} c_{\mathrm{crit}}\right) - \sqrt{\left(c - \frac{1}{2} c_{\mathrm{crit}}\right)^2 - 24(d_B - d_F)\tilde{h} y}, \tag{3.6}$$

which provides an all-order expression for the coupling constant renormalization. The wavefunction renormalization constants $C(j,\alpha_j)$ can, albeit using a mild assumption, be computed as well [6].

4. The BRST structure of strings from gauged WZW models

Consider a bosonic string consisting of a matter, a gravity or Liouville and a ghost sector. The matter sector is represented by its energy-momentum tensor T_m which generates the Virasoro algebra with central charge c_m. The gravity sector is realized in terms of a Liouville field φ_L, $\partial\varphi_L(z_1)\partial\varphi_L(z_2) = -z_{12}^{-2}$, with energy-momentum tensor T_L:

$$T_L = -\frac{1}{2}\partial\varphi_L\partial\varphi_L + \sqrt{\frac{25 - c_m}{12}}\partial^2\varphi_L, \tag{4.1}$$

[3] This formula has to be slightly generalized in the case that the embedded $sl(2)$ has a non-trivial centralizer in $\mathcal{G}$ [4].

which has central charge $c_L = 26 - c_m$. The energy-momentum tensor for the ghosts, $T_{gh} = -2b\partial c - (\partial b)c$, has central extension $c_{gh} = -26$. The total energy-momentum tensor $T = T_m + T_L + T_{gh}$ has central charge 0. The BRST current

$$J_{BRST} = c\left(T_m + T_L + \frac{1}{2}T_{gh}\right) + \alpha\partial\left(c\partial\varphi_L\right) + \beta\partial^2 c, \tag{4.2}$$

with

$$\alpha = -\frac{\sqrt{3}}{6}\left(\sqrt{1-c_m} + \sqrt{25-c_m}\right), \qquad \beta = \frac{1}{2}(1-\alpha^2), \tag{4.3}$$

has only regular terms in its OPE with itself. The total derivative terms in Eq. (4.2), wich have no influence on the BRST operator, have precisely been added to achieve this [7]. Calling $G_+ \equiv J_{BRST}$ and $G_- \equiv b$, one finds that the current algebra generated by T, G_+ and G_- closes, provided a $U(1)$ current U is introduced:

$$\begin{aligned}
&T(z_1)T(z_2) = 2z_{12}^{-2}T(z_2) + z_{12}^{-1}\partial T(z_2), \quad T(z_1)G_+(z_2) = z_{12}^{-2}G_+(z_2) + z_{12}^{-1}\partial G_+(z_2),\\
&T(z_1)G_-(z_2) = 2z_{12}^{-2}G_-(z_2) + z_{12}^{-1}\partial G_-(z_2),\\
&T(z_1)U(z_2) = -\frac{c_{N=2}}{3}z_{12}^{-3} + z_{12}^{-2}U(z_2) + z_{12}^{-1}\partial U(z_2),\\
&G_+(z_1)G_-(z_2) = \frac{c_{N=2}}{3}z_{12}^{-3} + z_{12}^{-2}U(z_2) + z_{12}^{-1}T(z_2),\\
&U(z_1)G_\pm(z_2) = \pm z_{12}^{-1}G_\pm(z_2), \qquad U(z_1)U(z_2) = \frac{c_{N=2}}{3}z_{12}^{-2},
\end{aligned} \tag{4.4}$$

where $U \equiv -bc - \alpha\partial\varphi_L$ is a modified ghost number current. Upon untwisting $T_{N=2} = T - \frac{1}{2}\partial U$, one gets the $N=2$ superconformal algebra with central extension $c_{N=2} = 6\beta$.

We now turn to the Hamiltonian reduction and show how to obtain the above from it. We want to identify G_+ with the BRST current and G_- with the Virasoro anti-ghost, so a single field instead of a composite. To achieve this, we have to consider a different grading, according to the eigenvalues of $\frac{1}{2}\mathrm{ad}_{e_0} + \mathrm{ad}_{u_0}$. We obtain for the gradings of the various currents:

	$E^{\mp}$	E^0	$E^{=}$	U^0	F^+	F^-	$\bar{F}^+$	$\bar{F}^-$
grade	1	0	-1	0	3/2	1/2	$-1/2$	$-3/2$

We follow the same procedure as before, but whenever we refer to the grading on the Lie algebra, we always imply it to be the grading induced by $\frac{1}{2}\mathrm{ad}_{e_0} + \mathrm{ad}_{u_0}$. Again we constrain the strictly negatively graded part of the algebra:

$$\Pi_{<0}J = \frac{\kappa}{2}\left(e_{=} + \psi\bar{g}_+\right). \tag{4.5}$$

The current $\bar{F}^{+}$ is a highest $sl(2)$ weight and will become the leading term of a conformal current. But on the same token, $\bar{F}^{+}$ has a negative grading, so it has to be constrained. Thus we need to constrain it in a non-singular way, *i.e.* by putting it equal to an auxiliary field which is inert under the gauge transformations. The action which reproduces the constraints is easily obtained:

$$\mathcal{S} = \kappa S^{-}[g] + \frac{1}{\pi x}\int str\, A(J - \frac{\kappa}{2}e_{=} - \frac{\kappa}{2}\Psi) + \frac{\kappa}{2\pi x}\int str \Psi\bar{\partial}\bar{\Psi}, \tag{4.6}$$

where

$$A = A^{\ddagger}e_{\ddagger} + A^{+}g_{+} + A^{-}g_{-}, \qquad \Psi = \psi\bar{g}_{+}, \qquad \bar{\Psi} = \bar{\psi}g_{-}. \tag{4.7}$$

The gauge invariance is parametrized by $\eta \in \Pi_{>0}sl(2|1)$ or $\eta = \eta^{\ddagger}e_{\ddagger} + \eta^{+}g_{+} + \eta^{-}g_{-}$:

$$\delta g = \eta g, \quad \delta A = \bar{\partial}\eta + [\eta, A], \quad \delta\bar{\Psi} = \eta^{-}g_{-}. \tag{4.8}$$

The combined requirements of gauge invariance and the existence of a non-degenerate highest weight gauge, requires the introduction of the field $\bar{\psi}$ conjugate to ψ. As before, the gauge choice is $A = 0$. Introducing ghosts $c = c^{\ddagger}e_{\ddagger} + \gamma^{+}g_{+} + \gamma^{-}g_{-} \in \Pi_{>0}sl(2|1)$ and anti-ghosts $b = b^{=}e_{=} + \bar{\beta}^{+}\bar{g}_{+} + \bar{\beta}^{-}\bar{g}_{-} \in \Pi_{<0}sl(2|1)$, we get the gauge fixed action:

$$\mathcal{S}_{gf} = \kappa S^{-}[g] + \frac{\kappa}{2\pi x}\int str\Psi\bar{\partial}\bar{\Psi} + \frac{1}{2\pi x}\int str b\bar{\partial}c, \tag{4.9}$$

and the nilpotent BRST charge

$$Q_{HR} = \frac{1}{4\pi i x}\oint str\left\{c\left(J - \frac{\kappa}{2}e_{=} - \frac{\kappa}{2}\Psi + \frac{1}{2}J_{gh}\right)\right\}. \tag{4.10}$$

Again, the affine symmetry of the WZW model breaks down to an $N = 2$ superconformal symmetry whose generators the generators of the cohomology $\mathcal{H}^{*}(\mathcal{A}, Q_{HR})$, where $\mathcal{A}$ is the algebra generated by $\{b, \hat{J} = J + J_{gh}, \psi, \bar{\psi}, c\}$. We are now in the position to use spectral sequence techniques, to compute the cohomology. One arrives at

$$\begin{aligned}
T_{N=2} &= \frac{2\kappa}{\kappa+1}\Big(\hat{E}^{\ddagger} + \psi\hat{F}^{-} - \frac{2}{\kappa}\hat{U}^{0}\hat{U}^{0} + \frac{2}{\kappa}\hat{E}^{0}\hat{E}^{0} \\
&\quad -\partial\hat{E}^{0} - \frac{1}{\kappa}\partial\hat{U}^{0} - \frac{\kappa+1}{4}\left(3\psi\partial\bar{\psi} + \partial\psi\bar{\psi}\right)\Big) \\
G_{+} &= \frac{2\kappa^{2}}{1+\kappa}\Big(\hat{F}^{+} + \hat{E}^{\ddagger}\bar{\psi} - \frac{2}{\kappa}(\hat{E}^{0} + \hat{U}^{0})\hat{F}^{-} + \hat{F}^{-}\bar{\psi}\psi + \partial\hat{F}^{-} \\
&\quad -\frac{(\kappa+1)(2\kappa+1)}{4\kappa}\partial^{2}\bar{\psi} + \frac{2}{\kappa}(\hat{E}^{0}\hat{E}^{0} - \hat{U}^{0}\hat{U}^{0})\bar{\psi} \\
&\quad -\partial(\hat{E}^{0} - \hat{U}^{0})\bar{\psi} - \frac{1+\kappa}{2}\psi\partial\bar{\psi}\bar{\psi} + \frac{2(1+\kappa)}{\kappa}\partial\bar{\psi}\hat{U}^{0}\Big) \\
G_{-} &= \psi \qquad\qquad U = -4\left(\hat{U}^{0} + \frac{\kappa}{4}\psi\bar{\psi}\right),
\end{aligned} \tag{4.11}$$

which satisfies the $N=2$ superconformal algebra with $c_{N=2} = -3(1+2\kappa)$. The Miura transform is again given by the algebra isomorphism which maps the currents in eq. (4.11) on their grade 0 part (the Ψ and $\bar{\Psi}$ fields have grade 0). This together with the OPE's $\hat{E}_0(z_1)\hat{E}_0(z_2) = -\hat{U}_0(z_1)\hat{U}_0(z_2) = (\kappa+1)/8\, z_{12}^{-2}$ and $\psi(z_1)\bar{\psi}(z_2) = 1/\kappa\, z_{12}^{-1}$ gives the desired realization of the $N=2$ algebra. Indeed, identifying $b \equiv \psi$, $c \equiv \kappa\bar{\psi}$, $\partial\varphi_L \equiv \sqrt{8/(\kappa+1)}\hat{U}_0$ and $\partial\varphi_m \equiv i\sqrt{8/(\kappa+1)}\hat{E}_0$, and

$$T_m = -\frac{1}{2}\partial\varphi_m\partial\varphi_m + i\frac{\kappa}{\sqrt{2(\kappa+1)}}\partial^2\varphi_m, \tag{4.12}$$

precisely reproduces, upon twisting, the non-critical string theory discussed at the beginning of this section with

$$c_m = 1 - 6\left(\sqrt{\kappa+1} - \frac{1}{\sqrt{\kappa+1}}\right)^2. \tag{4.13}$$

This program has been carried out in general [8]. The classification of all possible $sl(2|1)$ embeddings in super Lie algebras, yields all possible extensions of the $N=2$ superconformal algebra. The subset of embeddings of $sl(2|1)$ in $\mathcal{G}$ which allow for a stringy interpretation, are those for which the adjoint representation of $\mathcal{G}$ decomposes into $sl(2|1)$ irreps (b,j), with $b=0$ [4]. This is only possible for the following cases:

1. $sl(m|n)$
   A principal embedding of $sl(2|1)$ in $p\, sl(2j+1|2j) \oplus q\, sl(2j|2j+1)$, which in its turn is regularly embedded in $sl(m|n)$ with $p,\ q \geq 0$, $j \in \frac{1}{2}\mathbf{N}$, $m = p(2j+1)+2qj$ and $n = 2pj + q(2j+1)$.

2. $osp(m|2n)$
   The regular embedding of $osp(2|2)$ in $osp(m|2n)$.

3. $D(2,1,\alpha)$
   $osp(2|2)$ as a regular subalgebra of $D(2,1,\alpha)$.

After the reduction we are left with the $N=2$ superconformal currents together with a set of $N=2$ multiplets each of which generically contains four currents which yields some extension of the $N=2$ superconformal algebra. The currents fall into unconstrained $N=2$ multiplets each containing four currents, say $Y(x)$, $H_+(x)$, $H_-(x)$ and $Z(x)$ of conformal dimensions $h+1$, $h+1/2$, $h+1/2$ and h. Twisting amounts to replacing $Y(x)$ by $X(x) \equiv Y(x) + \frac{1}{2}\partial Z(x)$. If G_- and all fields of the H_- type are realized as single fields, something which is achieved by modifying the

[4] This requirement follows from considering the ghost number assignments in the resulting string theory. Typical $sl(2|1)$ irreps (b,j), $b \neq \pm j$, consist of 4 sl(2) irreps: $|j,b>$, $|j-1/2, b\pm 1/2>$ and $|j-1,b>$, where the second label denotes half the $u(1)$ eigenvalue. Atypical irreps $(\pm j, j)$ contain only 2 $sl(2)$ irreps. In the string theory, the $u(1)$ eigenvalue gets identified with the ghostnumber.

canonical $sl(2)$ grading by adding a multiple of the $u(1)$ charge to it, then we can view the system as a string theory. Indeed, one gets from $N = 2$ representation theory the following results,

$$Q = \frac{1}{2\pi i} \oint dz\, G_+(z), \tag{4.14}$$

is the BRST charge of the string, satisfying $Q^2 = 0$. The G_- current and all currents of the H_- type are the anti-ghosts. The total symmetry currents (matter + gravity + ghosts) are the energy-momentum tensor $T = T_{N=2} + \frac{1}{2}\partial U$ and the currents of the X type and they are indeed the BRST transform of the corresponding antighosts:

$$T = [Q, G_-]_+ \qquad\qquad X = [Q, H_-]_\pm. \tag{4.15}$$

The explicit construction of the gauge invariant action and its quantization has been performed in [8]. Modulo some complications mostly of a technical nature, the general discussion parallels the example given at the beginning of this section. The stringtheories which can be obtained this way are:

1. for $sl(2|1) \to p\, sl(2j+1|2j) \oplus q\, sl(2j|2j+1) \hookrightarrow sl(p(2j+1)+2qj|2pj+q(2j+1))$ one has that the matter sector of the string theory corresponds to the reduction:

$$sl(2) \to p\, sl(2j+1) \oplus q\, sl(2j+1) \hookrightarrow sl(p(2j+1)|q(2j+1)). \tag{4.16}$$

2. for $osp(2|2) \hookrightarrow osp(m|2n)$ the matter sector is now given by the reduction

$$sp(2) \hookrightarrow sp(2n) \hookrightarrow osp(m-2|2n). \tag{4.17}$$

In particular, we get that the first case covers W_n strings by choosing $j = (1/2)(n-1)$, $p = 1$ and $q = 0$ [9] and by choosing $m = N + 2$ and $n = 1$ in the second case, we get the N-extended superstrings [10]. At this point it is an interesting open question to find out which string theory corresponds to the reduction of $D(2, 1, \alpha)$. A priori, one would expect the $N = 2$ superstring. However, the BRST structure of $N = 2$ superstrings was explicitely studied in [10] where it was found that $N = 2$ superstrings were covered by $osp(4|2)$ or in other words $D(2, 1, \alpha = 1)$.

5. Discussion

We saw that embeddings of $sl(2)$ in (super) Lie algebras provide a powerful method to construct cft's and $2D$ gravity theories. The embedding completely determines the conformal symmetry. The precise realization of this symmetry depends on the grading, or equivalently the gauge group, chosen on the Lie algebra. Through the study of $sl(2|1)$ embeddings, one not only classifies extended $N = 2$ superconformal symmetries but using a particular grading , one gets the explicit construction of string

theories as well. The last result follows from the observation that the BRST strucure of a string theory is fully characterized by a twisted extended $N = 2$ superconformal symmetry.

Several open problems remain. Most of the work performed till now was done at the level of constructing actions, symmetry currents, BRST charges, ... Though it is a non-trivial result that these things can be obtained in an algebraic, almost algorithmic way (*e.g.* before the development of these methods, the only way to construct the BRST charge of strings was through trial and error, a method which in most cases led to formidable computer calculations), one would like to push this program further and obtain results on spectrum, correlation functions, ... Due to recent results in [12] where the structure of singular and subsingular vectors in affine Lie algebra Verma modules got unravelled, one hopes that at least partition functions become calculable.

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References

1. E. Witten, Comm. Math.Phys. **92** (1984) 455
2. A. M. Polyakov and P. B. Wiegman, Phys. Lett. **B141** (1984) 223
3. A. Bais, T. Tjin and P. van Driel, Nucl. Phys. **B357**, (1991) 632
4. A. Sevrin and W. Troost, Phys. Lett. **B315** (1993) 304
5. J. de Boer and T. Tjin, Comm. Math.Phys. **160** (1994) 317
6. J. de Boer and J. Goeree, Nucl.Phys. **B401** (1993) 348; A. Sevrin, K. thielemans and W. Troost, Nucl. Phys. **B407**, (1993) 459, Phys. Rev **D48** (1993) 2789
7. B. Gato-Rivera and A. M. Semikhatov, Phys. Lett. **B293** (1992) 72.
8. E. Ragoucy, A. Sevrin and P. Sorba, *Strings from $N = 2$ Gauged Wess-Zumino-Witten Models*, preprint ENSLAPP-555/95
9. M. Bershadsky, W. Lerche, D. Nemeschansky and N. P. Warner, Nucl. Phys. **B401** (1993) 304
10. A. Boresch, K. Landsteiner, W. Lerche and A. Sevrin, Nucl. Phys. **B436** (1995) 609
11. K. Landsteiner, W. Lerche and A. Sevrin, Phys. Lett. **B352** (1995) 286
12. P. van Driel and K. de Vos, *The Kazhdan-Lusztig conjecture for W Algebras*, preprint, hep-th/9508020